Diffeomorphisms, Noether Charges and Canonical Formalism in 2D Dilaton Gravity [†]

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Abstract

We carry out a parallel study of the covariant phase space and the conservation laws of local symmetries in two-dimensional dilaton gravity. Our analysis is based on the fact that the Lagrangian can be brought to a form that vanishes on-shell giving rise to a well-defined covariant potential for the symplectic current. We explicitly compute the symplectic structure and its potential and show that the requirement to be finite and independent of the Cauchy surface restricts the asymptotic symmetries.

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1 Introduction.

Great efforts have been developed recently on the study of two-dimensional dilaton gravity theories. The reason for this interest is that these theories serve as toy models in which we can develop and test techniques and methods to be further applied to more realistic (higher dimensional) gravity theories. Remarkably, the string inspired model (CGHS-model) of Ref. [1] (see also [2]) admits black hole solutions and, therefore, provides an interesting toy model to study black hole issues.

One of the aims of this paper is to study the reduced phase space of the CGHS-model. This point could be of great interest for a non-perturbative canonical quantization of the theory. Our work is based on the covariant phase-space formalism [3]-[7] and extends the results of a previous paper [8]. The covariant formalism has already been applied to the CGHS-model in Ref. [9, 10] although their results are valid for the case of a closed space only.

Moreover, for Lagrangians vanishing on-shell, the Noether's procedure can be incorporated, in a rather natural way, to the covariant canonical formalism. Therefore, we shall also study, in a parallel way, the covariant phase space and the conservation laws associated with diffeomorphism invariance. Our analysis will shed new insight on the controversy about the notion of mass in 2D dilaton gravity (see [11]-[13]).

In Section 2 we present briefly the covariant phase space formalism pointing out the fact that, for Lagrangians vanishing on-shell, the space of solutions can be endowed with a natural potential for the symplectic structure. The Noether charge technique is naturally incorporated in this scheme. In Section 3 we study in a systematic way the conservation laws associated with the diffeomorphism invariance and, in particular, with the asymptotic (Poincaré) symmetries of the CGHS model. In Section 4 we determine the symplectic potential of the CGHS model. The condition of having a well-defined potential (i.e. finite and independent of the Cauchy surface) will restrict the allowed asymptotic symmetries. The Lorentz symmetry break down and the spatial translation turns out to be a gauge-type transformation. This will permit to understand the results of Section 3. We shall also consider, in Section 5, the case of spherically symmetric $3+1$ Einstein gravity, which can also be regarded as a 2D dilaton gravity model (see [14] for a related perspective). Although the stringy and Schwarzschild black holes have the same canonical structure they differ in the form of the potential. As a byproduct, this accounts for the numerical factors in the Komar-type formulas for the mass in gravity models. We state our conclusions in Section 6.

2 Covariant phase space and conservation laws

Given a field theory with dynamical fields $\Psi^\alpha(x)$ and action $S = S(\Psi^\alpha(x))$, the phase space can be defined, in a covariant way, as the space of solutions of the classical equations of motion. The standard formula

$$\delta S(X^\alpha) = \int_{\mathcal{M}} \frac{\delta S}{\delta \Psi^\alpha} X^\alpha + \partial_\mu j^\mu(\Psi^\alpha, X^\beta) \tag{1}$$

can be interpreted now as the exterior derivative of S, on the covariant phase space, acting on a tangent vector X^α (which solves the linearized equations of motion). In contrast with the variational calculus which takes the variation X^α vanishing on the boundary of $\mathcal{M}$, it is now the first term of the r.h.s. of (1) which vanishes automatically. Therefore, the covariant phase space can be equipped with a presymplectic two-form

$$\omega = \int_\Sigma \delta j^\mu d\sigma_\mu \, , \tag{2}$$

where Σ is a Cauchy hypersurface and δ stands for the exterior derivative operator. Due to the fact that the symplectic current $\omega^\mu = \delta j^\mu$ is conserved, the presymplectic form (2) is, in general grounds, independent of the Cauchy surface with a suitable choice of boundary conditions.

From the above expression it is clear that the one-form

$$\theta = \int_\Sigma j^\mu d\sigma_\mu \tag{3}$$

could serve as a potential for the presymplectic form (2). However, j^μ is not, in general, conserved and hence θ is not well-defined.

Now, let us suppose that the presymplectic potential current j^α is itself conserved,

$$\partial_\alpha j^\alpha_{|_{sol}} = 0\,. \tag{4}$$

Then, for any field $X \sim \delta\Psi^a$ satisfying the linearized equations of motion, we will have that $J^\alpha_X = \mathrm{i}_X j^\alpha$ is a conserved current:

$$\partial_\alpha J^\alpha_{X\,|_{sol}} = 0\,. \tag{5}$$

What is the condition for a presymplectic potential current to be conserved? On solutions we have

$$\partial_\alpha j^\alpha_{|_{sol}} = \delta \mathcal{L}_{|_{sol}}\,. \tag{6}$$

Therefore, it is enough that the Lagrangian vanishes on the covariant phase space. In this situation the one-form (3) is well defined (with appropriate boundary conditions), $J^\alpha_X = j^\alpha(X)$ coincides with the Noether current and $\theta(X)$ is the corresponding Noether charge.

3 Energy-momentum conservation in the CGHS model.

The action of the CGHS model is:

$$S_{CGHS} \;=\; \frac{1}{2}\int_{\mathcal{M}} d^2x\sqrt{-g}\left[\mathrm{e}^{-2\Phi}(R + 4(\nabla\Phi)^2 + 4\lambda^2) - \frac{1}{2}(\nabla\phi_i)^2\right]\,. \tag{7}$$

By doing $\varphi = \mathrm{e}^{-\Phi}$ we obtain

$$S_{CGHS} \;=\; \frac{1}{2}\int_{\mathcal{M}} d^2x\sqrt{-g}\left[(R\varphi^2 + 4(\nabla\varphi)^2 + 4\lambda^2\varphi^2) - \frac{1}{2}(\nabla\phi_i)^2\right]\,, \tag{8}$$

which, for our purposes, is a form of the action more easy to deal with.

Now, it is convenient to define a new metric $\widehat{g}_{\nu\mu}$ by means of

$$g = \varphi^{-2}\widehat{g}\,, \tag{9}$$

in term of which the action takes a remarkably simpler form:

$$S_{CGHS} = \frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^{\,2}x\sqrt{-\widehat{g}}\left[\left(\hat{R}\varphi^2 + 4\lambda^2\right) - \frac{1}{2}(\nabla\phi_i)^2\right]\,, \tag{10}$$

The new variable $\widehat{g}$, which allows to eliminate the kinetic term in the action, also emerges in the gauge-theoretical formulation [15] of the theory, and in more general models [16].

The equations of motions are given by:

$$\widehat{R} = 0 \quad , \quad \widehat{\Box}\varphi^2 = 4\lambda^2 \quad , \quad \widehat{\Box}\phi_i = 0 \, , \tag{11}$$

$$\widehat{\nabla}_\mu \widehat{\nabla}_\nu \varphi^2 = \frac{1}{2}\widehat{\Box}\varphi^2 + \frac{1}{2}\left(\frac{1}{2}(\widehat{\nabla}\phi_i)^2\right)\widehat{g}_{\mu\nu} - \frac{1}{2}\widehat{\nabla}_\mu\phi_i\widehat{\nabla}_\nu\phi_i \, , \tag{12}$$

and, if we add a convenient total divergence to the action of the CGHS model in (10) we can easily bring it to a form vanishing on-shell

$$\widehat{S}_{CGHS} = \int_{\mathcal{M}} \widehat{\mathcal{L}}_{CGHS} = \frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^2x\sqrt{-\widehat{g}}\left[\left(\hat{R}\varphi^2 + 4\lambda^2 - \widehat{\nabla}_\alpha\widehat{\nabla}^\alpha\varphi^2\right) + \frac{1}{2}\phi_i\widehat{\Box}\phi_i\right] . \tag{13}$$

The symplectic potential associated to the above Lagrangian is:

$$\begin{aligned} \hat{j}^\alpha = & \frac{1}{2}\sqrt{-\widehat{g}}\Big[-\varphi^2\left(\widehat{g}^{\mu\nu}\widehat{\nabla}^\alpha\delta\widehat{g}_{\mu\nu} - \widehat{g}^{\mu\alpha}\widehat{\nabla}^\nu\delta\widehat{g}_{\mu\nu}\right) \\ & + \frac{1}{2}\widehat{\nabla}^\alpha(\varphi^2)\widehat{g}^{\mu\nu}\delta\widehat{g}_{\mu\nu} - \widehat{g}^{\mu\alpha}\widehat{\nabla}_\mu\delta(\varphi^2) \\ & - \frac{1}{2}(\widehat{\nabla}^\alpha\phi_i\delta\phi_i - \phi_i\widehat{\nabla}^\alpha\delta\phi_i) - \frac{1}{2}(\phi_i\widehat{\nabla}^\mu\phi_i)\widehat{g}^{\nu\alpha}\delta\widehat{g}_{\mu\nu} + \frac{1}{2}g^{\mu\nu}\delta g_{\mu\nu}(\frac{1}{2}\phi_i\widehat{\nabla}^\alpha\phi_i)\Big] \, . \end{aligned} \tag{14}$$

It can be shown by direct computation, and using the equations of motions, that the above symplectic potential is preserved actually.

The conserved current associated to a diffeomorphism generated by a vector field $X_f = f^\mu \frac{\partial}{\partial x^\mu}$, defined on the configuration space of the theory as

$$(\delta g)_{\mu\nu} = \nabla_\mu f_\nu + \nabla_\nu f_\mu \quad , \quad \delta\varphi = f^\mu\partial_\mu\varphi \, , \tag{15}$$

can be written in the form:

$$\begin{aligned} \widehat{J}^\alpha_f = & \frac{1}{2}\sqrt{-\widehat{g}}\Big\{\widehat{\nabla}_\mu[f^\mu\widehat{\nabla}^\alpha\varphi^2 - f^\alpha\widehat{\nabla}^\mu\varphi^2] + \widehat{\nabla}_\mu(\varphi^2[\widehat{\nabla}^\mu f^\alpha - \widehat{\nabla}^\alpha f^\mu]) \\ & + \frac{1}{2}\widehat{\nabla}_\mu(f^\mu\phi_i\widehat{\nabla}^\alpha\phi_i - f^\alpha\phi_i\widehat{\nabla}^\mu\phi_i)\Big\} \, . \end{aligned} \tag{16}$$

It has, therefore, the form of the divergence of an antisymmetric tensor and is, because of that, identically preserved (notice, however, that arriving at eq. (16) requires to use the equations of motion). The conserved charge associated to $\widehat{J}^\alpha_f$ can be made explicit by noticing that the divergence of an antisymmetric tensor $F^{\mu\nu}$ can be written, in 2D, as

$$\begin{aligned} \sqrt{-g}\nabla_\mu F^{\mu\nu} = & \sqrt{-g}\nabla_\mu\left[\frac{1}{2}(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha})F_{\alpha\beta}\right] = \\ = & \sqrt{-g}\nabla_\mu\left[\frac{1}{2}\frac{1}{g}\varepsilon^{\mu\nu}\varepsilon^{\alpha\beta}F_{\alpha\beta}\right] = \varepsilon^{\mu\nu}\partial_\nu K \, , \end{aligned} \tag{17}$$

with

$$K = -\frac{1}{2}\frac{1}{\sqrt{-g}}\varepsilon^{\alpha\beta}F_{\alpha\beta} \, . \tag{18}$$

Therefore,

$$\widehat{J}^\alpha_f = \varepsilon^{\alpha\beta}\partial_\beta\widehat{K} \, , \tag{19}$$

with

$$\widehat{K} = -\frac{1}{2}\frac{1}{\sqrt{-\widehat{g}}}\varepsilon^{\mu\nu}\left(f_\mu\widehat{\nabla}_\nu\varphi^2 + \varphi^2\widehat{\nabla}_\mu f_\nu + \frac{1}{2}f_\mu\phi_i\widehat{\nabla}_\nu\phi_i\right) \, . \tag{20}$$

In terms of the physical metric $g_{\mu\nu}$ the conserved current is given by

$$\begin{aligned} J_f^\alpha &= \frac{1}{2}\sqrt{-g}\nabla_\mu\left[\varphi^2(\nabla^\mu f^\alpha - \nabla^\alpha f^\mu) + \frac{1}{2}(f^\mu\phi_i\nabla^\alpha\phi_i - f^\alpha\phi_i\nabla^\mu\phi_i)\right] \\ &= \varepsilon^{\alpha\beta}\partial_\beta K\,, \end{aligned} \tag{21}$$

with the charge

$$K = -\frac{1}{2}\frac{1}{\sqrt{-g}}\varepsilon^{\mu\nu}\left(\varphi^2\nabla_\mu f_\nu + \frac{1}{2}f_\mu\phi_i\nabla_\nu\phi_i\right)\,. \tag{22}$$

It is interesting to compare (21) with Komar's formula for the conserved current in 4D [13, 17], and to notice that the presence of the matter term in (21) has its origin in the total divergence terms added to the Lagrangian. On the other hand, these total divergence terms in the Lagrangian are the reason why eq. (21) differs from other expressions for J_f^α given in the literature [18] and, as we will see, they contribute to make K finite, under appropriate asymptotic conditions.

From the above expressions it is not difficult to obtain, by choosing $f^\mu = \varepsilon^{\mu\nu}x_\nu + a^\mu$ and following the generalized Belinfante procedure [13], a symmetric energy-momentum pseudotensor for the CGHS model:

$$\Theta^{ab} = \frac{1}{2}\partial_\mu\partial_\nu\left(\sqrt{-g}\varphi^2\left[\eta^{ab}g^{\mu\nu} - \eta^{\mu b}g^{\nu a} - \eta^{\nu a}g^{\mu b} + \eta^{\mu\nu}g^{ab}\right]\right)\,. \tag{23}$$

On the other hand, in the absence of matter, any solution of the equations of motion can be brought, by means of a diffeomorphism, to the form

$$\mathrm{d}\,s^2 = -\left(\frac{m}{\lambda} - \lambda^2 x^+ x^-\right)^{-1}\mathrm{d}\,x^+\mathrm{d}\,x^- \quad , \quad \varphi^2 = \frac{m}{\lambda} - \lambda^2 x^+ x^-\,, \tag{24}$$

where x^+, x^- can be considered as the null Kruskal coordinates. The spacetime has four regions which can be characterised by the sign of the Kruskal coordinates. The asymptotic flat regions are characterised by $-\lambda^2 x^+ x^- > 0$. In the region I ($x^+ > 0$, $x^- < 0$) the metric can be written in a static asymptotically-flat form:

$$\mathrm{d}\,s^2 = -\left(1 + \frac{m}{\lambda}\mathrm{e}^{-2\lambda\sigma}\right)^{-1}\mathrm{d}\,\sigma^+\mathrm{d}\,\sigma^-\,, \tag{25}$$

$$\mathrm{e}^{-2\Phi} = \frac{m}{\lambda} + \mathrm{e}^{2\lambda\sigma}\,, \tag{26}$$

by means of the coordinate change

$$\lambda x^+ = e^{\lambda(\tau+\sigma)}\,, \tag{27}$$

$$\lambda x^- = -e^{-\lambda(\tau-\sigma)}\,. \tag{28}$$

In the other asymptotically flat region II ($x^+ < 0\,,\ x^- > 0$) the static metric can be achieved by the change

$$\lambda x^+ = -e^{\lambda(\tau+\sigma)}\,, \tag{29}$$

$$\lambda x^- = e^{-\lambda(\tau-\sigma)}\,. \tag{30}$$

If we calculate the energy of the basic solution of the CGHS model, eq. (25,26), by means of this E-M pseudotensor we will, surprisingly, not find any sensible result. In fact, the resulting expression is divergent and even do not involve the constant m. In the next sections, we will find the explanation for this result: the construction of a symmetrized E-M pseudotensor requires the theory to be invariant under asymptotic Lorentz transformations. We will show, however, that in order

to have a well defined physical theory, we can not allow pure-Lorentz asymptotic rotations.

Going back to eq. (21), the contribution to the conserved charge for the basic solution in (25-26) is:

$$K_I \;=\; \frac{1}{2}\left\{\left(\frac{m}{\lambda}+e^{2\lambda\sigma}\right)\partial_\sigma f^\tau+\left(\frac{m}{\lambda}+e^{2\lambda\sigma}\right)\partial_\tau f^\sigma+2mf^\tau\right\}_{|\sigma\to+\infty}\;, \tag{31}$$

where the subindex I refers to the region in which the above current has been evaluated. With the asymptotic fall-off conditions

$$\begin{aligned} e^{2\lambda\sigma}\partial_\sigma f^\tau &\overset{\sigma\to\infty}{\sim} 0\,,\\ e^{2\lambda\sigma}\partial_\tau f^\sigma &\overset{\sigma\to\infty}{\sim} 0\,, \end{aligned} \tag{32}$$

the Noether charge associated with the Killing time translation ($f^\tau \overset{\sigma\to\infty}{\sim} 1$) is $K_I = m$. Terms like $\lambda e^{2\lambda\sigma} f^\tau$, that would appear in the expression for the Noether charge had we started with Lagrangian (8), cancel out in (31). It is just the Lagrangian (13) which gives directly the finite terms only. The reason is that the Noether charge (31) can be seen as the result of contracting the presymplectic potential with the infinitesimal diffeomorphism X associated with the asymptotic time translation (in region I). Both quantities are well defined in the covariant phase space, as we will see in the next section. The charges associated with the asymptotic spatial translations and Lorentz transformations are zero and divergent, respectively.

Moreover we also want to stress that the Noether charge (31) just gives the mass of the black hole without the discrepant factor $\frac{1}{2}$, as happens in the Komar's formula for energy in General Relativity. We shall also understand this fact in the context of the canonical formalism.

4 Canonical structure and asymptotic symmetries of the CGHS model.

Let us begin our analysis of the canonical structure of the CGHS model by writing the general classical solution of the theory without matter. It is well known that any solution is equivalent under diffeomorphisms to the solution

$$\mathrm{d}\,\hat{s}^2 = -\mathrm{d}\,x^+\mathrm{d}\,x^- \quad , \quad \varphi^2 = \frac{m}{\lambda} - \lambda^2 x^+ x^-\,. \tag{33}$$

The solutions are characterized by an unique diffeomorphism invariant parameter, m, and therefore the variable canonically conjugate to m should be "hidden" in the group of diffeomorphisms. The situation is somewhat similar to the trivial example of the free particle. Any solution is equivalent, under the Galileo group, to the one with the particle lying at rest and, therefore, the canonical degrees of freedom of the system are found in the symmetry (Galileo) group.

Our aim now is to find the degrees of freedom of the theory that are "hidden" in the group of diffeomorphisms. To this end we shall compute explicitly the two-form (2) (more precisely, the potential one-form (3)). This requires to adjust the boundary condition adequately for the potential form to be finite and independent of the spacelike Cauchy surface. Therefore, we shall assume the metric $g_{\mu\nu}$ to be flat at spatial infinity with a specific fall-off behaviour.

Let us apply a general diffeomorphism to the basic solution (33). We find

$$\mathrm{d}\,\hat{s}^2 \;=\; -\mathrm{d}\,P\mathrm{d}\,M\,, \tag{34}$$

$$\varphi^2 \;=\; \frac{m}{\lambda} - \lambda^2 PM\,, \tag{35}$$

where P and M are two arbitrary functions $P, M \; : \mathcal{M} \rightarrow R$; $x^+ = P(\tau,\sigma)$, $x^- = M(\tau,\sigma)$.

We have

$$\widehat{g}_{\mu\nu} = -\frac{1}{2}\left(\partial_\mu P \partial_\nu M + \partial_\mu M \partial_\nu P\right) , \tag{36}$$

$$\sqrt{-\widehat{g}} = \frac{1}{2}\varepsilon^{\alpha\beta}\partial_\alpha P \partial_\beta M , \tag{37}$$

$$\widehat{g}^{\mu\nu} = -\frac{1}{(\sqrt{-\widehat{g}})^2}\varepsilon^{\mu\alpha}\varepsilon^{\nu\beta}\widehat{g}_{\alpha\beta} , \tag{38}$$

$$\widehat{\Gamma}_{\mu\alpha\beta} = -\frac{1}{2}\left[\partial_\alpha\partial_\beta P \partial_\mu M + \partial_\alpha\partial_\beta M \partial_\mu P\right] , \tag{39}$$

$$\widehat{\Gamma}^\mu_{\alpha\beta} = \frac{1}{2\sqrt{-\widehat{g}}}\varepsilon^{\mu\nu}\left[\partial_\alpha\partial_\beta P \partial_\nu M - \partial_\alpha\partial_\beta M \partial_\nu P\right] . \tag{40}$$

Obviously we also have

$$\widehat{\nabla}_\alpha \widehat{\nabla}_\beta M = 0 \quad , \quad \widehat{\nabla}_\alpha \widehat{\nabla}_\beta P = 0 \quad , \quad \forall a, b , \tag{41}$$

and, therefore, for the metric parametrized as in eq. (36):

$$\begin{aligned} \delta\widehat{g}_{\alpha\beta} &= -\frac{1}{2}\left(\partial_\beta M \delta\partial_\alpha P + \partial_\alpha P \delta\partial_\beta M\right. \\ &\quad \left. + \partial_\beta P \delta\partial_\alpha M + \partial_\alpha M \delta\partial_\beta P\right) \\ &= -\frac{1}{2}\left[\widehat{\nabla}_\alpha(\widehat{\nabla}_\beta M \delta P + \widehat{\nabla}_\beta P \delta M) + \widehat{\nabla}_\beta(\widehat{\nabla}_\alpha P \delta M + \widehat{\nabla}_\alpha M \delta P)\right] \quad (42) \\ &= \widehat{\nabla}_\alpha h_\beta + \widehat{\nabla}_\beta h_\alpha , \quad (43) \end{aligned}$$

where the one-form h_μ is given by:

$$h_\mu = -\frac{1}{2}\left(\widehat{\nabla}_\mu P \delta M + \widehat{\nabla}_\mu M \delta P\right) \quad \Rightarrow \quad h^\alpha = -\frac{1}{2}\widehat{g}^{\alpha\mu}\left(\widehat{\nabla}_\mu P \delta M + \widehat{\nabla}_\mu M \delta P\right) . \tag{44}$$

We can easily see that, with the one-form h^α defined above, we can write as well:

$$\widehat{\nabla}_\mu \delta\varphi^2 = \widehat{\nabla}_\mu (h^\alpha \widehat{\nabla}_\alpha \varphi^2) \quad , \quad \forall \mu . \tag{45}$$

So, to get the symplectic potential for the general solution given in (34-35), it is enough to replace in eq. (16) the diffeomorphism f^μ by the quantities h^μ as defined in (44).

The symplectic potential will therefore be given by the divergence of an anti-symmetric tensor (K is now a one-form)

$$\widehat{j}^\alpha = \varepsilon^{\alpha\mu}\partial_\mu K , \tag{46}$$

and the symplectic form will be a pure-boundary term, thus implying that the theory has a finite number of degrees of freedom.

4.1 Conditions of flatness at spatial infinity.

The condition for the metric to be flat at spacelike infinity means:

$$g_{\mu\nu} = -\frac{1}{2}\frac{\partial_\mu P \partial_\nu M + \partial_\mu M \partial_\nu P}{\bar{m} - \lambda^2 PM} \xrightarrow{\text{spacelike}} \eta_{\mu\nu} , \tag{47}$$

where $\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\bar{m} = \frac{m}{\lambda}$.

In region I ($P \equiv x^+ > 0$ and $M \equiv x^- < 0$) we can make

$$\lambda P = \mathrm{e}^{\lambda C} \quad , \quad -\lambda M = \mathrm{e}^{\lambda R} \, . \tag{48}$$

Using (47) and because of $-PM \overset{\text{spacelike}}{\longrightarrow} +\infty$, we arrive at

$$\mathrm{d}\, C \, \mathrm{d}\, R \overset{\text{spacelike}}{\longrightarrow} -\mathrm{d}\, \tau^2 + \mathrm{d}\, \sigma^2 \, , \tag{49}$$

or, what is the same,

$$\begin{aligned} &\dot{C}\dot{R} \overset{\text{spacelike}}{\longrightarrow} -1 \, , \\ &\dot{C}R' + C'\dot{R} \overset{\text{spacelike}}{\longrightarrow} 0 \, , \\ &C'R' \overset{\text{spacelike}}{\longrightarrow} 1 \, , \end{aligned} \tag{50}$$

requirements whose solution can be written in the form:

$$C(\tau, \sigma) \;=\; \alpha\sigma^+ + A + U(\tau, \sigma) \, , \tag{51}$$

$$R(\tau, \sigma) \;=\; -\frac{1}{\alpha}\sigma^- - B + V(\tau, \sigma) \, , \tag{52}$$

where α, A and B are real numbers, $\sigma^+ = \tau + \sigma$, $\sigma^- = \tau - \sigma$, and

$$U, V \overset{\text{spacelike}}{\longrightarrow} 0 \, . \tag{53}$$

The interpretation of (51-52) on the light of (49) is obvious: the only allowed diffeomorphisms (C, R) are those that asymptotically are Poincaré transformations in the coordinates τ, σ. Surprisingly we will find additional constraints on the asymptotic transformations in the computation of the (on-shell) symplectic potential.

4.2 Symplectic potential.

The symplectic current potential is given by

$$\widehat{j}^\alpha = \varepsilon^{\alpha\mu}\partial_\mu \widehat{K} \, , \tag{54}$$

with $\widehat{K}$, formally, given by (20).

The first consequence of the above formulas is that the symplectic potential reads as

$$\theta = \int_\Sigma \partial_\mu \widehat{K} \mathrm{d}\, x^\mu = \widehat{K}(i^0_R) - \widehat{K}(i^0_L) \, , \tag{55}$$

where Σ is an arbitrary Cauchy surface (see Fig. I). We have to stress that Σ is not required to intersect the bifurcation point of the horizon as it was in Ref. [8]. The point now is to show that the one-form K can have well defined values in the right and left spatial infinities. In fact we shall find that not all the asymptotic Poincaré transformations are permitted in order to have a well defined result for θ (i. e., independent of the Cauchy surface).

Replacing the "diffeomorphism" h^μ by its expression in eq. (44) we find after a bit of algebra:

$$\begin{aligned} \widehat{K}(P, M, m) \;=\; & -\frac{1}{2}\lambda^2 \left(P\delta M - M\delta P\right) \\ & -\frac{1}{4\sqrt{-\widehat{g}}}\lambda^2 PM\varepsilon^{\lambda\rho}\left(\partial_\rho P\delta\partial_\lambda M + \partial_\rho M\delta\partial_\lambda P\right) \\ & +\frac{\bar{m}}{4\sqrt{-\widehat{g}}}\varepsilon^{\lambda\rho}\left(\partial_\rho P\delta\partial_\lambda M + \partial_\rho M\delta\partial_\lambda P\right) \, , \end{aligned} \tag{56}$$

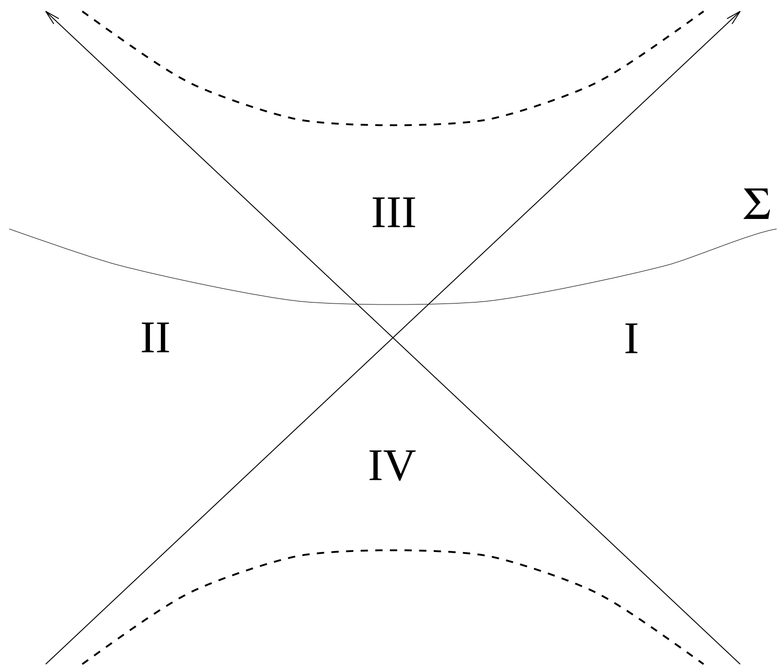

Figure 1: Kruskal diagram for a black hole spacetime. Σ is an arbitrary Cauchy surface.

and, after having made use of (48), we find

$$\begin{aligned}\widehat{K}(C,R,m) &= -\frac{1}{4\chi}\lambda^2 PM\varepsilon^{\lambda\rho}\left(\partial_\rho C\delta\partial_\lambda R+\partial_\rho R\delta\partial_\lambda C\right)\\ &+\frac{\bar{m}}{4\chi}\varepsilon^{\lambda\rho}\left(\partial_\rho C\delta\partial_\lambda R+\partial_\rho R\delta\partial_\lambda C\right)\\ &-\frac{m}{2}\delta(R-C)\,,\end{aligned}\tag{57}$$

where χ is given by:

$$\chi=\frac{1}{2}\varepsilon^{\lambda\rho}\partial_\lambda C\partial_\rho R \overset{\text{spacelike}}{\longrightarrow} 1\,.\tag{58}$$

And after the replacements in (51,52) we find

$$\begin{aligned}\widehat{K} &= \frac{1}{4\chi}(\bar{m}-\lambda^2PM)\varepsilon^{\lambda\rho}\left(\partial_\rho U\delta\partial_\lambda V+\partial_\rho V\delta\partial_\lambda U\right)\\ &+\frac{1}{2\chi}(\bar{m}-\lambda^2PM)(\frac{1}{\alpha}\delta\partial_+U+\alpha\delta\partial_-V)\\ &-\frac{1}{2\chi}(\bar{m}-\lambda^2PM)(\partial_+U\,\delta\frac{1}{\alpha}+\partial_-V\,\delta\alpha)\\ &-\frac{m}{2}\,\delta(V-U)\\ &+\frac{1}{2\chi}(\bar{m}-\lambda^2PM)\frac{2}{\alpha}\delta\alpha\\ &+\frac{m}{2}\delta(\frac{1}{\alpha}\sigma^-+\alpha\sigma^+)\\ &+m\,\delta\left(\frac{A+B}{2}\right)\,.\end{aligned}\tag{59}$$

It is easy to realize from the last expression above that for the symplectic potential to be finite and independent of the spacelike Cauchy surface (i.e, independent

of τ), requires first that $\alpha = 1$. That is to say, the Lorentz transformations are not allowed. So that, we are left with:

$$\begin{aligned}\widehat{K} &= \frac{1}{4\chi}(\bar{m} - \lambda^2 PM)\varepsilon^{\lambda\rho}\left(\partial_\rho U \delta\partial_\lambda V + \partial_\rho V \delta \partial_\lambda U\right) \\ &\quad + \frac{1}{2\chi}(\bar{m} - \lambda^2 PM)(\delta\partial_+ U + \delta\partial_- V) \\ &\quad + m\,\delta\left(\frac{A+B}{2}\right).\end{aligned} \tag{60}$$

Moreover, to find a finite resulting expression for (60) we have to require an appropriate asymptotic fall-off for the functions U and V. From a close inspection of eq. (60), and taking into account the asymptotic behaviour of $-PM$, it is not difficult to realize that the most natural requirement in order to have a sensible reduced phase space is

$$\begin{aligned} e^{2\lambda\sigma}\dot{U}, e^{2\lambda\sigma}\dot{V} &\overset{\sigma\to\infty}{\sim} 0\,, \\ e^{2\lambda\sigma}U', e^{2\lambda\sigma}V' &\overset{\sigma\to\infty}{\sim} 0\,. \end{aligned} \tag{61}$$

Therefore we have arrived at:

$$\widehat{K}(i^0_R) = m\,\delta\left(\frac{A+B}{2}\right), \tag{62}$$

where $\frac{A+B}{2} \equiv f(i^0_R)$ is the Killing time translation at right spatial infinity.

In the other asymptotically flat region $x^+ = P(\tau,\sigma) < 0$, $x^- = M(\tau,\sigma) > 0$, we should write

$$-\lambda P = e^{\lambda C} \quad , \quad \lambda M = e^{\lambda R} \tag{63}$$

instead of (48), where the asymptotic flatness requires that (the asymptotic Lorentz transformation has already been neglected)

$$\begin{aligned} C(\tau,\sigma) &= \tau + \sigma + A + U(\tau,\sigma)\,, \\ R(\tau,\sigma) &= -(\tau - \sigma) - B + V(\tau,\sigma)\,, \end{aligned} \tag{64}$$

$$U, V \overset{\sigma\to\infty}{\sim} 0\,. \tag{65}$$

Proceeding in the same way as in the region I we obtain

$$\widehat{K}(i^0_L) = m\,\delta\left(\frac{A+B}{2}\right), \tag{66}$$

where now $\frac{A+B}{2} \equiv f(i^0_L)$ stands for the Killing time translation at left infinity.

Taking into account (62) and (66) we obtain the final expression for the symplectic potential

$$\theta = m\,\delta\left(f(i^0_R) - f(i^0_L)\right). \tag{67}$$

4.3 Diffeomorphisms in the presence of matter.

When matter is present, the procedure applied above is much more complicated. This is so because we would not be able to write the symplectic form as a pure boundary term. The model has an infinite number of degrees of freedom and, because of that, the symplectic form has, unavoidably, a bulk term. Intuitively we expect, however, that diffeomorphisms should be "almost" pure gauge. In the covariant formalism, this means that the presymplectic two-form (2) should be degenerated along the directions that corresponds to the gauge transformations of

the theory. We can arrive at this result by contracting the symplectic two-form with the generator of a diffeomorphism:

$$(\delta g)_{\mu\nu} = \nabla_\mu f_\nu + \nabla_\nu f_\mu \quad , \quad \delta\varphi = f^\mu \partial_\mu \varphi \quad , \quad \delta\phi_i = f^\mu \partial_\mu \phi_i \,. \tag{68}$$

The only linearized equation of motion which is not trivial to obtain is:

$$\widehat{\nabla}_\mu \delta\widehat{\Gamma}^\mu_{\alpha\beta} - \widehat{\nabla}_\alpha \delta\widehat{\Gamma}^\mu_{\beta\mu} = 0 \,. \tag{69}$$

and, after a long computation, we arrive at:

$$\mathrm{i}_{X_f}\, \delta j^\alpha = \partial_\lambda T^{\lambda\alpha} \,, \tag{70}$$

i.e. $\mathrm{i}_{X_f}\omega$ is a pure boundary term, with

$$\begin{aligned} T^{\lambda\alpha} = -T^{\alpha\lambda} \quad = \quad & \frac{1}{2}\sqrt{-\widehat{g}}\Big\{\varphi^2 \Big[-\delta \log \sqrt{-\widehat{g}}(\widehat{\nabla}^\lambda f^\alpha - \widehat{\nabla}^\alpha f^\lambda) \\ & \qquad +(\delta g^{\mu\alpha}\widehat{\nabla}_\mu f^\lambda - \delta g^{\mu\lambda}\widehat{\nabla}_\mu f^\alpha) \\ & \qquad +(f^\lambda \widehat{g}^{\mu\nu}\delta\widehat{\Gamma}^\alpha_{\mu\nu} - f^\alpha g^{\mu\nu}\delta\widehat{\Gamma}^\lambda_{\mu\nu}) \\ & \qquad +(f^\nu \widehat{g}^{\mu\alpha}\delta\widehat{\Gamma}^\lambda_{\mu\nu} - f^\nu g^{\mu\lambda}\delta\widehat{\Gamma}^\alpha_{\mu\nu}) \\ & \qquad +(f^\alpha \widehat{g}^{\mu\lambda}\delta\widehat{\Gamma}^\nu_{\mu\nu} - f^\lambda g^{\mu\alpha}\delta\widehat{\Gamma}^\nu_{\mu\nu})\Big] \\ & -\delta\varphi^2(\widehat{\nabla}^\lambda f^\alpha - \widehat{\nabla}^\alpha f^\lambda) \\ & +2(f^\alpha \widehat{\nabla}^\lambda \delta\varphi^2 - f^\lambda \widehat{\nabla}^\alpha \delta\varphi^2) \\ & +(f^\alpha \widehat{\nabla}_\mu \varphi^2 \delta g^{\lambda\mu} - f^\lambda \widehat{\nabla}_\mu \varphi^2 \delta g^{\alpha\mu}) \\ & +(f^\alpha \widehat{\nabla}^\lambda \phi_i \delta\phi_i - f^\lambda \widehat{\nabla}^\alpha \phi_i \delta\phi_i)\Big\} \,. \end{aligned} \tag{71}$$

It is convenient now to rewrite the expressions above in terms of the physical metric, which has a better behaviour at spacelike infinity. We find:

$$\begin{aligned} \frac{2}{\sqrt{-g}}T^{\lambda\alpha} \quad = \quad & \varphi^2 \big\{-\delta \log \sqrt{-g}\left(\nabla^\lambda f^\alpha - \nabla^\alpha f^\lambda\right) \\ & \quad -\delta \log \sqrt{-g}\left(f^\alpha \nabla^\lambda \log \varphi^2 - f^\lambda \nabla^\alpha \log \varphi^2\right) \\ & \quad +\left(\delta g^{\mu\alpha}\nabla_\mu f^\lambda - \delta g^{\mu\lambda}\nabla_\mu f^\alpha\right) \\ & \quad -\frac{1}{2}\left(\delta g^{\mu\alpha}\nabla_\mu \log \varphi^2 f^\lambda - \delta g^{\mu\lambda}\nabla_\mu \log \varphi^2 f^\alpha\right) \\ & \quad +\frac{1}{2}\left(\nabla^\alpha \log \varphi^2 f_\mu \delta g^{\mu\lambda} - \nabla^\lambda \log \varphi^2 f_\mu \delta g^{\mu\alpha}\right) \\ & \quad -\delta \log \varphi^2 \left(\nabla^\lambda f^\alpha - \nabla^\alpha f^\lambda\right) \\ & \quad +2\left(f^\alpha \nabla^\lambda \delta \log \varphi^2 - f^\lambda \nabla^\alpha \delta \log \varphi^2\right)\big\} \\ & +\left(f^\alpha \nabla^\lambda \phi_i \delta\phi_i - f^\lambda \nabla^\alpha \phi_i \delta\phi_i\right) \,. \end{aligned} \tag{72}$$

If we take, as it appears the most natural, boundary conditions such that $\phi_i \overset{\text{spacelike}}{\longrightarrow} 0$, the expressions above indicates clearly that the analysis of the model with matter reduces itself to the case without matter. Therefore, the contribution of diffeomorphisms to the reduced phase space of the theory is the same when there are matter fields as when there are not. For instance, if we take into account that $\varphi^2 \overset{\text{spacelike}}{\sim} e^{2\lambda\sigma}$, we see that the leading term in (72) behaves as $-\delta \log \varphi^2 \varphi^2 \left(\nabla^\lambda f^\alpha - \nabla^\alpha f^\lambda\right)$. The finiteness of this term implies $\varepsilon^{\mu\nu}\partial_\mu f_\nu \overset{\text{spacelike}}{\longrightarrow} 0$, thus forbidding as symmetries of the theory those diffeomorphisms that are asymptotically Lorentz transformations.

5 Symplectic potential of Schwarzschild black holes.

The symplectic current potential of general relativity in vacuum is given by

$$j^{\alpha}=\frac{1}{16\pi}\sqrt{-g}\left(g^{\mu\nu}\delta\Gamma^{\alpha}_{\mu\nu}-g^{\mu\alpha}\delta\Gamma^{\nu}_{\mu\nu}\right)\ , \tag{73}$$

and due to the Hilbert-Einstein Lagrangian vanishes on-shell the current (73) is conserved. In this section we shall work out the symplectic potential associated with the Schwarzschild black hole solutions. Instead of starting with the basic solution and acting on it with a general diffeomorphism we shall assume that the relevant asymptotic symmetry is the Killing time translation. Therefore we can write the general solution in regions I and II as follows

$$ds^2=-\left(1-\frac{2m}{r}\right)d(t+f(t,r))^2+\left(1-\frac{2m}{r}\right)^{-1}dr^2+r^2\left(d\theta^2+\sin^2\theta d\varphi^2\right)\ . \tag{74}$$

In addition, we shall choose the Cauchy surface in such a way that it connects the spatial infinities through the asymptotically flat regions I and II.

The symplectic potential is the integral of an exact three-form and, therefore, it receives contribution from the two-spheres $S^2_{R,L}$ at infinity only

$$\begin{aligned}\theta \quad = \quad & \frac{1}{16\pi}\int_{S^2_R} d\Omega\ \sin\theta\left[-\frac{r(r-2m)}{1+\dot{f}}f'\,\delta f+r(r-2m)\,\delta f'\right.\\ & \left.\qquad -2f\,\delta m+2m\,\delta f\right]\\ & -\frac{1}{16\pi}\int_{S^2_L} d\Omega\sin\theta\left[-\frac{r(r-2m)}{1+\dot{f}}f'\,\delta f+r(r-2m)\,\delta f'\right.\\ & \left.\qquad -2f\,\delta m+2m\,\delta f\right]\ .\end{aligned} \tag{75}$$

To obtain a well-defined result we have to assume the following fall-off behaviour:

$$r^2 f',\dot{f}\overset{r\to\infty}{\sim}0\ . \tag{76}$$

With the prescribed fall-off the integral (75) turns out to be

$$\theta=\frac{1}{2}\left[m\,\delta\left(f(i^0_R)-f(i^0_L)\right)-\left(f(i^0_R)-f(i^0_L)\right)\ \delta m\right]\ . \tag{77}$$

6 Conclusions and final comments.

On the light of the result of Secs. 3-4 we observe that the asymptotic fall-off behaviour of the diffeomorphisms entering in the symplectic potential (32) are similar to that required to have a well-defined Noether charge (61). This is a consequence of the closed relationship between the canonical formalism and the Noether theorem outlined in Sec. 2.

Using the covariant phase space picture we have determined the canonical structure of the CGHS model in the absence of matter, and the character of the asymptotic symmetries, without any a priori assumption on the dilaton asymptotic behaviour. The requirements made in 4.1-2 on the metric are enough to arrive at a clear result. The difference of Killing time translations at spatial infinities turns out to be the conjugate variable to the black hole mass. The asymptotic spatial translations are "gauge"-type symmetries: they decouple in the symplectic potential

and leads to trivial Noether charge. The asymptotic Lorentz transformation breaks down (it cannot be permitted to have a well-defined symplectic form) and leads to a divergent Noether charge. This results are closely related. On general grounds, the action of a Lorentz transformation gives linear momentum to the system. In the CGHS model it breaks down and, therefore, the linear momentum vanishes identically, in accord with the "gauge" nature of the spatial translations for the model. This provides an explanation for the failure of the symmetric energy-momentum pseudotensor. The definition of this quantity requires the theory to be invariant under asymptotic Lorentz transformations and we have shown that this is not the case for the CGHS model.

As a byproduct of our study we also provide an explanation of the well-known factor 2 in the Komar formula for the mass in General Relativity. Although both the stringy and Schwarzschild black holes have the same symplectic structure

$$\omega = \delta\theta = \delta m \wedge \delta\left(f(i^0_R) - f(i^0_L)\right) \ , \tag{78}$$

they differ in the form of the symplectic potential. For the CGHS black hole the potential contains only the term with $\delta\left(f(i^0_R) - f(i^0_L)\right)$

$$\omega_{CGHS} = m\,\delta\left(f(i^0_R) - f(i^0_L)\right) \ . \tag{79}$$

The corresponding Noether charge associated with a (right) asymptotically Killing time translation is just the black hole mass

$$\theta_{CGHS}\left(\frac{\partial}{\partial f(i^0_R)}\right) = m\,. \tag{80}$$

In the case of Schwarzschild black hole the symplectic potential contains a term with δm as well. So that the Noether charge cannot coincide exactly with the mass. Since the potential is symmetric in m and $f(i^0_R) - f(i^0_L)$

$$\theta_{Sch} = \frac{1}{2}\left(m\,\delta\left(f(i^0_R) - f(i^0_L)\right) - \left(f(i^0_R) - f(i^0_L)\right)\,\delta m\right) \ , \tag{81}$$

the Noether charge is actually one half of the mass

$$\theta_{Sch}\left(\frac{\partial}{\partial f(i^0_R)}\right) = \frac{m}{2}\,. \tag{82}$$

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